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Some properties of a Lagrangian Wiener–Hermite expansion

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Wiener proposed that the turbulent velocity be expanded in Hermite functionals of a Gaussian white noise random function advected by the fluid. This paper describes the mechanics of converting his suggestion into a computable model, and assesses its range of validity as an approximation for incompressible, homogeneous, and isotropic turbulence. The terms retained are a linear term, $\mathbf{v}^1$, representing an arbitrary Gaussian velocity, and a quadratic term, $\mathbf{v}^2$, representing a non-Gaussian contribution to the velocity needed for energy transfer. The requirement that advection by the dependent velocity $\mathbf{v} = \mathbf{v}^1 + \mathbf{v}^2$ does not alter the statistics of the base necessitates a further truncation of the base to antisymmetric quadratic basis elements. Realizability of any statistics of $\mathbf{v}$ is common to all Wiener–Hermite expansions. The projected equations for the Lagrangian expansion conserve energy by non-linear interaction, preserve the inviscid Gaussian equipartition ensemble, and are invariant to random Galilean transformations. Numerical calculations with an approximate form of these equations reveal that irreversible relaxation to the inviscid equipartition solution is not a property of the Lagrangian model, and that the rapid convergence advanced as the original motivation for studying Wiener–Hermite expansions does not survive closure by truncation. The dynamics of the model is not inconsistent with the existence of an inertial range. A simple numerical search routine failed to produce a solution corresponding to such an equilibrium ensemble.

Wiener (1958, pp. 118–128) proposed a simple Lagrangian theory of turbulence in which the turbulent velocity fields are expanded in Hermite functionals of a Gaussian white noise random function advected by the fluid. The significance of Wiener's ideas was recognized in a series of papers by Siegel & Meecham (1959), Meecham & Siegel (1959, 1964), Imamura, Meecham & Siegel (1965), Siegel, Imamura & Meecham (1965) and Meecham & Jeng (1968) in which was developed a fixed base expansion applicable to fluid turbulence. They and their students, Su (1967) and Kahng (1968), applied the theory to the Burgers model and in approximate form to fluid turbulence. Despite the optimistic tone of Saffman's (1968) introduction to this version of the theory, it has serious flaws which have been pointed out by Orszag & Bissonnette (1967) and Crow & Canavan (1967).

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Some of the flaws which render the stationary base expansion invalid as an approximation to high Reynolds number turbulence are overcome by the expansion in an advected base which Wiener originally proposed. This paper presents the equations for Wiener's Lagrangian expansion, examines their analytical and numerical predictions, and assesses their range of validity as an approximation for incompressible, homogeneous, and isotropic turbulence. The model guarantees certain realizability and consistency properties which have eluded some other straightforward theories. It offers a novel solution to the closure problem encountered in treating non-linear problems statistically, but one which leads to difficulties in numerical calculations.

A number of mathematical problems are avoided by using a wave-vector space representation and treating quantities defined on the discrete array of wave vectors

$$\mathbf{k} = \frac{2\pi}{L}\mathbf{m}; \quad m_i = 0, \pm 1, \pm 2, \ldots, \tag{1}$$

corresponding to flows cyclic in cubes of side L. In this representation Gaussian white noise is a complex random vector field so defined (Canavan & Leith 1968) as to have the two point joint statistics

$$\left.\begin{aligned} \langle n_i(\mathbf{k})\, n_j(\mathbf{k}')\rangle &= \delta_{ij}\,\delta(\mathbf{k}+\mathbf{k}'), \\ \delta(\mathbf{0}) = 1; \quad \delta(\mathbf{k}) &= 0, \quad \mathbf{k} \neq \mathbf{0}, \end{aligned}\right\} \tag{2}$$

in terms of which all higher moments of the noise can be written. The linear and quadratic Hermite polynomials

$$\left.\begin{aligned} h^1_i(\mathbf{k},t) &= n_i(\mathbf{k},t), \\ h^2_{ij}(\mathbf{k},\mathbf{m},t) &= n_i(\mathbf{k},t)\,n_j(\mathbf{m},t) - \delta_{ij}\,\delta(\mathbf{k}+\mathbf{m}) \end{aligned}\right\} \tag{3}$$

can then be used to write the expansion of the velocity

$$\left.\begin{aligned} v_i(\mathbf{k},t) &= v^1_i(\mathbf{k},t) + v^2_i(\mathbf{k},t) \\ &= K^1_{ij}(\mathbf{k},t)\,h^1_j(\mathbf{k},t) + \sum_{\mathbf{k}'+\mathbf{k}''=\mathbf{k}} K^2_{ijk}(\mathbf{k}',\mathbf{k}'',t)\,h^2_{jk}(\mathbf{k}',\mathbf{k}'',t), \end{aligned}\right\} \tag{4}$$

which is appropriate to homogeneous turbulence. In this expansion the randomness of the velocity, and part of its time dependence, are carried by the noise base. The non-random kernels in the expansion, K^1 and K^2, determine directly all simultaneous statistics of the velocity. They are taken to have the isotropic tensor symmetries (Batchelor 1953, pp. 40–54)

$$K^1_{ij}(\mathbf{k}) = K^1_{ji}(\mathbf{k}) = K^1_{ij}(-\mathbf{k}), \quad K^2_{ijk}(\mathbf{k},\mathbf{m}) = -K^2_{ijk}(-\mathbf{k},-\mathbf{m}).$$

Incompressibility dictates that

$$k_i\,K^1_{ij}(\mathbf{k}) = 0, \quad (k_i + m_i)\,K^2_{ijk}(\mathbf{k},\mathbf{m}) = 0.$$

The linear term describes an arbitrary Gaussian velocity field. The quadratic term represents a non-Gaussian contribution to the velocity needed for energy transfer. The truncation of the expansion to two terms is dictated by practical computability. The use of an advected base does not, of course, alter the realizability property common to all Wiener–Hermite expansions (Meecham & Jeng 1968).

The expansion is Lagrangian in that each sample noise function $\mathbf{n}(\mathbf{k}, t)$ is constrained to evolve under the wave-vector space advection equation

$$\dot{n}_i(\mathbf{k},t) = -ik_m \sum_{\mathbf{k}'+\mathbf{k}''=\mathbf{k}} n_i(\mathbf{k}',t)\, v_m(\mathbf{k}'',t) \tag{5}$$

with $\mathbf{v}$ given by the expansion of (4). Thus, the base is imbedded in and moves with the fluid. Since the simple statistics of the noise motivates its choice as the basis for the expansion in the first place, it is essential that the time dependence of each noise function does not destroy those statistics. Canavan & Leith (1968) show that this requirement is satisfied if the additional antisymmetry

$$K^2_{jim}(\mathbf{k},\mathbf{m}) = -K^2_{jmi}(\mathbf{k},\mathbf{m}) \tag{6}$$

is imposed on K^2. This corresponds to a further truncation of the base which retains quadratic elements of the form

$$g^2_{ij}(\mathbf{k},\mathbf{m}) = \tfrac{1}{2}[h^2_{ij}(\mathbf{k},\mathbf{m}) - h^2_{ji}(\mathbf{k},\mathbf{m})]. \tag{7}$$

For such a second kernel, advection of the noise by the statistically dependent velocity of (4) does not alter the white Gaussian statistics of the noise ensemble. It should be noted that a truncation to antisymmetric quadratic basis elements produces a null theory of the Burgers turbulence.

Substitute the expansion of the velocity, (4), into the incompressible Navier–Stokes equations and take moments with h^1 and g^2 to arrive at non-random evolution equations for the kernels in the expansion

$$\dot{K}_{ij}(\mathbf{k}) + \nu k^2 K_{ij}(\mathbf{k}) + A_{ij}(\mathbf{k}) = C_{ij}(\mathbf{k}), \tag{8}$$

$$A_{ij}(\mathbf{k}) = -2ik_m \epsilon_{rjp} \int d\mathbf{m} K_{mp}(\mathbf{m})\, L_{ir}(\mathbf{k}-\mathbf{m},\mathbf{m}),$$

$$C_{ij}(\mathbf{k}) = -2i\epsilon_{rjp} P_{imn}(\mathbf{k}) \int d\mathbf{m} K_{mp}(\mathbf{m})\, L_{nr}(\mathbf{k},-\mathbf{m}),$$

$$2\dot{L}_{ij}(\mathbf{m},\mathbf{n}) + 2\nu k^2 L_{ij}(\mathbf{m},\mathbf{n}) + D_{ij}(\mathbf{m},\mathbf{n}) + E_{ij}(\mathbf{m},\mathbf{n}) = F_{ij}(\mathbf{m},\mathbf{n}) + G_{ij}(\mathbf{m},\mathbf{n}),\ \mathbf{m}+\mathbf{n}=\mathbf{k}, \tag{9}$$

$$E_{iq}(\mathbf{m},\mathbf{n}) = \tfrac{1}{2} i\epsilon_{qjk} k_m [K_{ik}(\mathbf{k}) K_{mj}(\mathbf{n}) + K_{ij}(\mathbf{k}) K_{mk}(\mathbf{m})],$$

$$F_{iq}(\mathbf{m},\mathbf{n}) = \tfrac{1}{2} i\epsilon_{qjk} P_{imn}(\mathbf{k}) K_{mk}(\mathbf{m}) K_{nj}(\mathbf{n}),$$

$$D_{iq}(\mathbf{m},\mathbf{n}) = 2i\epsilon_{qrs} \int d\mathbf{l}\,(k_m - l_m)\, L_{is}(\mathbf{k}-\mathbf{l},\mathbf{l})\,[L_{mr}(\mathbf{n},-\mathbf{l}) - L_{mr}(\mathbf{m},-\mathbf{l})],$$

$$G_{iq}(\mathbf{m},\mathbf{n}) = 2iP_{imn}(\mathbf{k})\, \epsilon_{qrs} \int d\mathbf{l} L_{ms}(\mathbf{m},\mathbf{l})\, L_{nr}(\mathbf{n},-\mathbf{l}),$$

where $\quad P_{ijk}(\mathbf{k}) = k_j P_{ik}(\mathbf{k}) + k_k P_{ij}(\mathbf{k}), \quad P_{ij}(\mathbf{k}) = \delta_{ij} - k_i k_j / k^2.$

The terms A, D and E which contain convolutions arise from the motion of the base. These equations differ from those of Canavan & Leith (1968) only by the replacement of K^2 by the isotropic pseudotensor L^2

$$L^2_{im}(\mathbf{m},\mathbf{n}) = \tfrac{1}{2}\epsilon_{mjk} K^2_{ijk}(\mathbf{m},\mathbf{n}), \tag{10}$$

the dropping of superscripts from K^1 and L^2, and the replacement of sums by integrals in the limit required for strict consistency with the assumption of isotropy.

The equations for the Lagrangian expansion exhibit several of the consistency properties discussed by Orszag & Kruskal (1968), Orszag (1970). In particular, they conserve energy through non-linear interaction, preserve the inviscid Gaussian equipartition ensemble and are invariant to random Galilean transformations. Energy conservation, for $\nu = 0$, follows by differentiating the expression for the spectral energy density

$$\begin{aligned} E(\mathbf{k}) &= \tfrac{1}{2}\langle v_i(\mathbf{k})\, v_i(-\mathbf{k})\rangle \\ &= \tfrac{1}{2}\langle v_i^1(\mathbf{k})\, v_i^1(-\mathbf{k})\rangle + \tfrac{1}{2}\langle v_i^2(\mathbf{k})\, v_i^2(-\mathbf{k})\rangle \\ &= E^1(\mathbf{k}) + E^2(\mathbf{k}) \end{aligned} \tag{11}$$

with respect to time, substituting for $\dot{K}$ and $\dot{L}$ from (8) and (9), summing over $\mathbf{k}$, and using

$$2\int d\mathbf{m} L_{ij}(\mathbf{m}, \mathbf{k}-\mathbf{m})\, E_{ij}(\mathbf{m}, \mathbf{k}-\mathbf{m}) = K_{ij}(\mathbf{k})\, A_{ij}(\mathbf{k}),$$

$$2\int d\mathbf{k} \int d\mathbf{m} L_{ij}(\mathbf{m}, \mathbf{k}-\mathbf{m})\, F_{ij}(\mathbf{m}, \mathbf{k}-\mathbf{m}) = \int d\mathbf{k} K_{ij}(\mathbf{k})\, C_{ij}(\mathbf{k}),$$

$$\int d\mathbf{m} L_{ij}(\mathbf{m}, \mathbf{k}-\mathbf{m})\, D_{ij}(\mathbf{m}, \mathbf{k}-\mathbf{m}) = 0 = \int d\mathbf{k} \int d\mathbf{m} L_{ij}(\mathbf{m}, \mathbf{k}-\mathbf{m})\, G_{ij}(\mathbf{m}, \mathbf{k}-\mathbf{m}),$$

to show that

$$\int d\mathbf{k}\,[\dot{E}^1(\mathbf{k}) + \dot{E}^2(\mathbf{k})] = 0.$$

The Gaussian equipartition ensemble (Kraichnan 1958) has the Wiener–Hermite expansion

$$K_{ij}(\mathbf{k}) = P_{ij}(\mathbf{k}), \quad L_{ij}(\mathbf{m}, \mathbf{n}) = 0. \tag{12}$$

For this choice of kernels (9), upon manipulation, yields $E = F$, so that $\dot{L}$ vanishes and this equilibrium ensemble is a stationary solution of the Lagrangian model.

Statistical Galilean invariance, the invariance of the dynamics of a turbulence theory to the inclusion of random uniform advecting velocities, has been discussed by Kraichnan (1964) and Orszag & Kruskal (1968) as a prerequisite to consistency with Kolmogorov's scaling arguments. Gaussian, $\mathbf{v}^1(\mathbf{0})$, and non-Gaussian, $\mathbf{v}^2(\mathbf{0})$, random Galilean transformations would be represented in the dynamics of the Lagrangian model by terms involving $K_{ij}(\mathbf{0})$ and $L_{ij}(\mathbf{k}, -\mathbf{k})$ respectively. Inspection of the $\dot{K}$ equation (8) shows, however, that the terms involving $\mathbf{v}^1(\mathbf{0})$ cancel between A and C and that the term in C involving $\mathbf{v}^2(\mathbf{0})$ vanishes by isotropy and incompressibility. Similarly, inspection of the $\dot{L}$ equation (9) shows that E and F cancel when $\mathbf{m}$ or $\mathbf{n} = 0$, so that $\mathbf{v}^1(\mathbf{0})$ makes no contribution to $\dot{L}$, and that the $\mathbf{1} = \mathbf{n}$ and $\mathbf{1} = \mathbf{m}$ terms of D cancel with the $\mathbf{1} = -\mathbf{m}$ and $\mathbf{1} = \mathbf{n}$ terms of G, so that $\mathbf{v}^2(\mathbf{0})$ makes no contribution. The use of a base imbedded in the fluid removes, as expected, the effects of any random uniform velocity from the dynamics of the Lagrangian expansion.

The equations which result from deleting D and G from the $\dot{L}$ equation (9)

$$\dot{K}_{ij}(\mathbf{k}) + \nu k^2 K_{ij}(\mathbf{k}) + A_{ij}(\mathbf{k}) = C_{ij}(\mathbf{k}), \tag{13}$$

$$2\dot{L}_{ij}(\mathbf{m}, \mathbf{n}) + 2\nu k^2 L_{ij}(\mathbf{m}, \mathbf{n}) + E_{ij}(\mathbf{m}, \mathbf{n}) = F_{ij}(\mathbf{m}, \mathbf{n}), \tag{14}$$

form a simpler set sharing the invariance and consistency properties of the Lagrangian model. Their solutions illustrate two important properties of time-dependent solutions of the full set. The first is that irreversible relaxation to the inviscid Gaussian equipartition solution (Kraichnan 1958, Orszag 1970) is not a property of either the simpler closure or the Lagrangian model. Modes perturbed from their equipartition levels oscillate about them with a frequency related to advective effects. This non-relaxation of departures from equipartition is demonstrated in numerical calculations with the simpler set but should not be qualitatively altered by reintroducing D and G. They are quadratic in perturbation quantities and hence legitimately ignored in this situation.

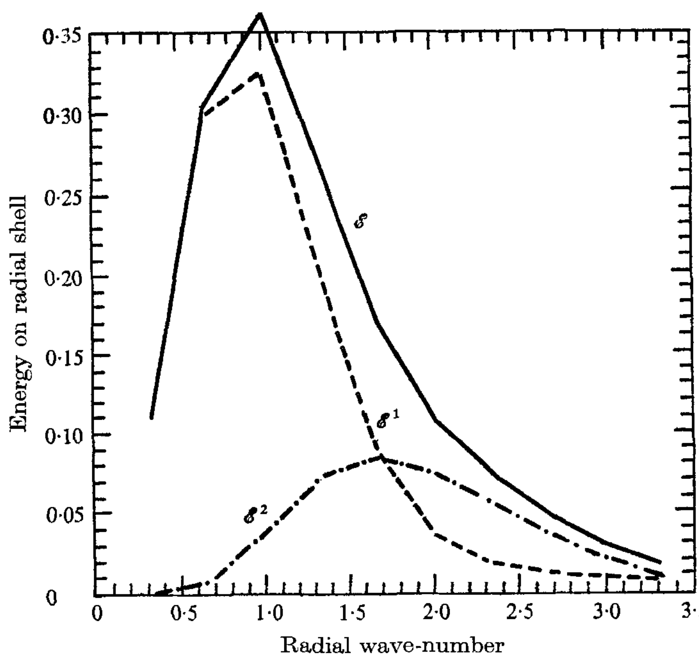

FIGURE 1. Distributions of $E^n(kL, t = 1{\cdot}5L/U)$, normalized by LU^2, which evolve from Gaussian initial conditions of equation (15) under equations (13) and (14).

The second feature demonstrated by solutions of the simpler set is that the rapid convergence advanced (Meecham & Jeng 1968) as the original motivation for studying Wiener–Hermite expansions is lost in truncation. From the initial conditions

$$E(k,t=0) = E^1(k,0) = 4\pi k^2 E^1(\mathbf{k},0) = 2\pi^{-\frac{1}{2}} LU^2 (kL)^2 \exp(-(kL)^2), \quad (15)$$

the equations produce the energy spectra of $\mathbf{v}^1$ and $\mathbf{v}^2$ shown in figure 1 at $t = 1{\cdot}5\ L/U$. Since $E^2(k) > E^1(k)$ for $k > 2$, $\mathbf{v}^2(\mathbf{k})$ cannot be regarded as a perturbation on a dominant Gaussian $\mathbf{v}^1(\mathbf{k})$ for such modes. The cause of this divergence is the lack of a mechanism in the $\dot{L}$ equation (14) for relaxing triple correlations built up by non-equipartition initial conditions. Thus, the form

$$L_{ij}(\mathbf{m},\mathbf{n},t) \approx \tfrac{1}{2}t[E_{ij}(\mathbf{m},\mathbf{n},0) - E_{ij}(\mathbf{m},\mathbf{n},0)], \quad (16)$$

which is certainly valid for $t \ll L/U$ also holds approximately for $t \gtrsim L/U$, predicting correctly the shape of $E^2(k)$ shown in figure 1 as well as the fact that

$$E^2(k,t) > E^1(k,t) \quad \text{for} \quad t > \pi/kU, \tag{17}$$

which is observed in computations. These calculations ignore the relaxation of correlations by molecular viscosity. Its inclusion, however, is only capable of preventing divergences for Reynolds numbers $\leqslant 2$.

The unphysical behaviour which invalidates quantitative predictions of the simpler closure is not necessarily met in the Lagrangian model. When $E^2(k) \approx E^1(k)$, D and G in the $\dot{L}$ equation (9) are as large as E and F, so that their inclusion may eliminate the secular growth of L. This demonstrates, however, that even the Lagrangian model will not retain the ordering

$$E^1(\mathbf{k}) \gg E^2(\mathbf{k}) \gg E^3(\mathbf{k}) \gg \ldots$$

implicitly assumed in truncating the expansion of $\mathbf{v}$ to two terms.

Since the unique feature of the expansion under discussion is its Lagrangian nature, it is appropriate to test the model's high Reynolds number dynamics in a Kolmogorovian inertial range. The time independent, self-similar forms of the kernels appropriate to such an equilibrium ensemble

$$K_{ij}(a\mathbf{k}) = a^{-x}K_{ij}(\mathbf{k}), \quad L_{ij}(a\mathbf{m}, a\mathbf{n}) = a^{-(x+\frac{3}{2})}L_{ij}(\mathbf{m},\mathbf{n}), \quad x = \tfrac{11}{6} \tag{18}$$

greatly simplify this investigation. While the inertial range form of K is completely determined by its isotropic form and scaling properties, that of the second kernel is not. An approximate form $\bar{L}$

$$\bar{L}_{ij}(\mathbf{m}, \mathbf{k}-\mathbf{m}) = \tau[F_{ij}(\mathbf{m}, \mathbf{k}-\mathbf{m}) - E_{ij}(\mathbf{m}, \mathbf{k}-\mathbf{m})], \quad \tau \propto k^{x-\frac{5}{2}}, \tag{19}$$

is obtained by formally integrating the $\dot{L}$ equation (9) from Gaussian initial conditions over a time τ which produces the scaling of (18), following a procedure due to Orszag & Kruskal (1968). Using these forms of K and $\bar{L}$ to perform the integrals in the evolution equations (8) and (9) at an inertial range wavevector $\mathbf{k}$ leads to no divergences, demonstrating that the model is not inconsistent with the existence of an inertial range. It also verifies, however, that the approximate form $\bar{L}$ given by (19) does not produce either $\dot{K} = 0$ or $\dot{L} = 0$. A stationary form of L is sought by a variational procedure. Suitable reduction tensors R and S are constructed (appendix) so that the inertial range scalar generating functions of L may be obtained by contraction

$$\psi(m,n,1) = R_{ij}(\mathbf{m},\mathbf{n})\,L_{ij}(\mathbf{m},\mathbf{n}), \tag{20}$$

$$\phi(m,n,1) = S_{ij}(\mathbf{m},\mathbf{n})\,L_{ij}(\mathbf{m},\mathbf{n}), \tag{21}$$

for $\mathbf{m}+\mathbf{n} = \mathbf{k}$, $|\mathbf{k}| = 1$, as are generating tensors W and X so that L may be reconstructed from them

$$L_{ij}(\mathbf{m},\mathbf{n}) = W_{ij}(\mathbf{m},\mathbf{n})\,\psi\left(\frac{m}{k},\frac{n}{k},1\right)k^{-(x+\frac{9}{2})} + \left[X_{ij}(\mathbf{m},\mathbf{n})\,\phi\left(\frac{m}{k},\frac{n}{k},1\right) - X_{ij}(\mathbf{n},\mathbf{m})\,\phi\left(\frac{n}{k},\frac{m}{k},1\right)\right]k^{-(x+\frac{3}{2})}, \tag{22}$$

$\mathbf{m}+\mathbf{n} = \mathbf{k}$, any k. The scalar generating functions are defined by their values on a rectangular array of points in the m, n plane, values at intermediate points needed to perform the integrals in (8) and (9) being found by linear interpolation. By systematically varying the values of the functions on the array of points, a form of L is sought which minimizes $\dot{L}$ over the array subject to $\dot{K} = 0$. This part of the investigation was unsuccessful. No choice of parameters produced a solution significantly better than the trivial one. No claims are made that the results were insensitive to the crude differencing involved in specifying L by two to six numbers, but such a sensitivity is in itself a failure for a statistical theory.

In conclusion, the use of advected Gaussian white noise as the basis for a Lagrangian expansion has been shown to produce a computable model of turbulence with several desirable consistency properties, but numerical evidence makes it appear unlikely that the Lagrangian expansion as formulated here can be regarded as a useful approximation for turbulence.

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Appendix

Proudman & Reid (1954) give the most general isotropic tensor funciton of two wave-vector arguments and three tensor subscripts. Owing to the symmetries expressed in (4) and (6) together with those dictated by isotropy and incompressibility it may be written as

$$K^2_{ijk}(\mathbf{m},\mathbf{n}) = P_{il}(\mathbf{m}+\mathbf{n})[\psi(m,n,k)(n_l n_j m_k - n_l m_j n_k) - \phi(m,n,k)(m_k\delta_{lj} - m_j\delta_{lk}) + \phi(n,m,k)(n_k\delta_{lj} - n_j\delta_{lk})],$$

where ψ and ϕ are arbitrary functions of m, n, and $k = |\mathbf{k}| = |\mathbf{m}+\mathbf{n}|$ which satisfy $\psi(m,n,k) = -\psi(n,m,k)$. Define

$$N_j(\mathbf{m},\mathbf{n}) = \epsilon_{jrs} m_r n_s,$$
$$M_i(\mathbf{m},\mathbf{n}) = \epsilon_{irs} N_r(\mathbf{m},\mathbf{n}) k_s,$$

and in terms of them

$$W_{ij}(\mathbf{m},\mathbf{n}) = -(1/k^2) M_i(\mathbf{m},\mathbf{n}) N_j(\mathbf{m},\mathbf{n}),$$
$$X_{ij}(\mathbf{m},\mathbf{n}) = P_{is}(\mathbf{m}+\mathbf{n})\epsilon_{sjr} m_r,$$
$$R_{ij}(\mathbf{m},\mathbf{n}) = -(M_i N_j + N_i M_j)/N^4,$$
$$S_{ij}(\mathbf{m},\mathbf{n}) = -N_i n_j/N^2,$$

then (20)–(22) follow from a simple computation. Contract R and S with the approximate $\bar{L}$ of (19) to obtain

$$\bar{\psi}(m,n,k) = 0,$$

$$\bar{\phi}(m,n,k) = k^{x-\frac{5}{2}}\left[(U(m)-U(k))\,U(n)+\frac{\mathbf{m}\cdot\mathbf{n}}{m^2}(U(n)-U(k))\,U(m)\right],$$

where $U(k) \propto k^{-x}$ is the scalar generating function for K^1. This result shows that ϕ is excited by terms quadratic in K^1 in the $\dot{L}$ equation (9), and that ψ is then produced by the terms quadratic in L. Only $\bar{\phi}$ is used in examining the equations for the Lagrangian expansion for divergences in an inertial range. Essential features of this examination are the cancellation between F and E for $m \ll k$ which leads to a form of $\bar{L}$ scaling less strongly on m than that which would evolve from either term alone, and a similar cancellation between G and D which prevents the alteration of this scaling by the terms quadratic in $\bar{L}$.

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